Relational or Calculational Thinking: Students Solving Open Number Equivalence Problems

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Student transition from arithmetic to algebraic reasoning has been acknowledged as an essential but problematical process. Previous research has highlighted the difficulties in shifting students towards relational thinking when solving equivalence problems. This paper reports on an investigation into students' use of relational thinking to solve equivalence problems after they have been in classrooms where specific focus has been on developing flexible, efficient computational strategies. The results reveal that most students used computational strategies to solve the equivalence problems rather than relational strategies. Many of the common errors, students made reflected a lack of understanding of the equal sign.

Introduction

Over the past decade, there has been increased focus, both in national and international research and reform efforts, on the teaching and learning of algebraic reasoning (e.g., Irwin, & Britt, 2005; Knuth, Stephens, McNeil, & Alibabi, 2006; Ministry of Education (MOE), 2006; National Council of Teachers of Mathematics (NCTM), 2000). Such emphasis has arisen from growing acknowledgment of the insufficient algebraic understandings students develop during schooling and the role this has in denying them access to potential educational and employment prospects (Knuth et al., 2006). In response, there have been significant curricular reforms designed to support students making the transition from arithmetic to algebra (Freiman & Lee, 2004; Kaput, 1999). One approach has been to promote students' ability to work flexibly with numbers. By developing the students' computational strategies it is claimed that their "structural thinking can then be exploited to develop their understanding of algebra" (Hannah, 2006, p. 1). This paper explores the strategies that students within the age band of 9 to 13 years old used when solving open number equivalence problems. It also investigates the common errors students made when solving these problems due to their lack of understanding of the equal sign. This study adds to previous research of student understanding of equivalence through the analysis of the common errors the students made.

Concepts of equivalence and understanding of the equal sign are essential to algebraic understanding (Freiman & Lee, 2004; Knuth et al., 2006). The foundations of the transition from arithmetic to algebraic reasoning requires that students are able to abstract key concepts including those associated with equivalence and relations. For students to abstract their structural numerical reasoning across to algebraic reasoning it is necessary they understand the equal sign relationally as an equivalence symbol meaning the "same as" (Knuth et al., 2006; McNeil & Alibabi, 2005). The seminal research of Kieran (1981) illustrated that students often have inadequate understanding of the equality symbol. Recent research continues to show that many primary and middle school students lack deep understanding of the equal sign (Carpenter, Franke, & Levi, 2003; Falkner, Levi, & Carpenter, 1999; Knuth et al., 2006; McNeil & Alibabi, 2005). Students with limited understanding of the equal sign view it as an indication of where to put the answer, or alternatively, equate the symbol with doing something – a left to right action or carrying out an operation (Carpenter et al., 2003; Rivera, 2006; Warren & Cooper, 2005). Inadequate understanding of the equality symbol can lead to difficulties for students solving symbolic expressions and equations (Kieran, 1981; Knuth et al., 2006). Furthermore, a limited understanding of the equality symbol can make the transition to algebra difficult for students (McNeil & Alibabi, 2005).

Errors made by students when solving open number equivalence problems reflect their understanding of the equal sign. Freiman and Lee (2004) demonstrated that open number sentence problems in the form of a + b = d + c involving a blank in the last two positions consistently caused difficulties across grade levels. Carpenter et al. (2003) argue that students' errors in solving open number sentence problems are errors of syntax. Students erroneously interpret the rules for how the equal sign is utilised. For example, when solving 9 + 6 = __ + 5, students may put 15 in the blank space considering that the equal sign is an indication to put an answer. Alternatively other students may put 20 in the blank space. These students overgeneralize the property of addition and assume the sequence of symbols in the number sentence is unimportant.

Understanding of the equality symbol as a sign of relational equivalence is a hallmark of the transition between arithmetic to algebraic thinking (Carpenter et al., 2005). Students with a relational view of the equal sign view it as a symbol of equivalence or quantitative sameness. Relational understanding enables students to solve open number sentence equivalence problems such as 8 + 4 = __ + 5 successfully (Falkner et al., 1999). However, within this group of students who understand the equal sign as a symbol of equivalence further distinctions can be made. These distinctions are between students who use computational forms of thinking or those who use relational forms of thinking to solve open number sentence problems.

Stephens (2006) defines relational thinking as dependent on whether children are "able to see and use possibilities of variation between numbers in a number sentence" (p. 479). Students who are able to use relational thinking to solve open number sentence problems consider the expressions on both sides of the equal sign. They are able to solve the problem by using the relation between both expressions without carrying out a calculation. In contrast, students who use computational thinking view the numbers on each side of the equal sign as representing separate calculations. These students perform a calculation to solve open number sentence problems (Carpenter et al., 2003; Stephens, 2006). Students who successfully use relational thinking to solve equivalence problems are also able to identify the direction in which the missing number will change, in order to maintain equivalence. Direction of variation in equivalence problems involving addition is different from those problems that involve subtraction. Stephens maintains that this can cause further difficulties for students.

Warren and Pierce (2004) propose that the difficulties that students encounter may be due to differentiation in requirements for algebraic reasoning and arithmetical reasoning. Some researchers have suggested that classroom mathematics experiences in the early years of schooling are the basis for many problems. This is particularly when emphasis is placed on computation and students are presented with the equal sign as a signal to carry out a calculation (e.g., Carpenter et al., 2003; Knuth et al., 2006; Warren & Pierce, 2004). Warren (2003) also argues that there may be potential problems associated with current reform shifts that focus on a need for number sense and identification of computational patterns. She maintains that these need to be balanced with explicit abstracting of arithmetic structures.

Advocates of mathematics curriculum reform initiatives have suggested teaching algebra and arithmetic as an integrated strand across the curriculum (e.g., Carpenter et al., 2003; NCTM, 2000; MoE, 2006). This approach focuses on building early algebraic thinking through focusing on students' informal knowledge and numerical reasoning. Teachers who use this approach provide students with learning situations which challenge their notions of equality and encourage them to think about relations. This supports students' transition from computation to relational thinking (Carpenter et al., 2003). Stephens' (2006) comparison of two Australian schools found that students exhibited higher levels of relational thinking to solve open number equivalence problems within a school that had a specialist mathematics teacher who explicitly focused on teaching of relational approaches. However, acquiring understanding of equivalence and developing relational thinking is acknowledged as a complex and difficult task and one which necessitates substantial time and explicit teacher attention (Carpenter et al., 2003; Freiman & Lee, 2004).

Method

This study was exploratory in nature and used a qualitative case study design. The aim of the study was to explore student understanding of the equal sign and equivalence. In particular, the study addresses the following research questions.

- What strategies do students use to solve open number equivalence problems?
- What errors are commonly made by students when solving open number equivalence problems?

Participants

The participants were 361 primary and intermediate school students (37 Year 5 students aged 9-10; 47 Year 6 students aged 10-11; 145 Year 7 students aged 11-12; 132 Year 8 students aged 12-13). The study was conducted at a New Zealand urban primary school. The students came from a predominantly middle socio-economic home environment. They were primarily from a European New Zealand ethnic grouping (67%), with students of Maori ethnic grouping (5%), Pacific Island ethnic grouping (10%), Asian ethic grouping (7%), and Indian ethnic grouping (11%).

The school was in its third year of participating in the New Zealand Numeracy Project and algebra had been taught as a separate strand from the number (arithmetic) strand.

Data Collection

The students were given a pen and paper questionnaire derived from a questionnaire developed by Stephens (2006). This consisted of equivalence balance problems with missing numbers and a question about the equal sign. The questionnaire was completed by each individual in regular class time and adequate time was provided to complete it. The students were advised that the questionnaire was not a test but a way to find out how students would solve the problems.

This study reports on the students' responses to the following sets of open number equivalence problems. All the problems were presented in the form of

$a + b = c + d$ or $a – b = c – d$.

Each set of questions began with the words: "Write a number in each of the boxes to make a true statement. Explain your working."

Table 1
Sets of Open Number Equivalence Problems

Group A	Group B	Group C
23 + 15 = 26 + __	39 – 15 = 41 – __	746 – 262 + __ = 747
73 + 49 = 72 + __	99 – __ = 90 – 59	746 + __ – 262 = 747
43 + __ = 48 + 76	104 – 45 = __ – 46	
__ + 17 = 15 + 24		

Data Analysis

Data analysis used the scoring rubric devised by Stephens (2006), which categorised students thinking using a five-point scale. This scale categorised students' responses according to whether they used arithmetical thinking or different levels of relational thinking. Further to this categorisation, responses were grouped into three categories according to students' stability in using different types of thinking to solve the various sets of problems. These were: stable arithmetic thinkers, the students who used only arithmetic strategies; stable relational thinkers, the students who used only relational strategies; and the unstable relational thinkers, students who used a mixture of relational and arithmetic strategies.

The data set was then re-analysed to identify common error types exhibited across the four year levels. In particular, incorrect responses, which indicated a lack of understanding of the equal sign, were identified and analysed. Common erroneous responses were grouped into categories identified in Freiman and Lee's (2004) study. These included: direct sum, responses when the blank was in the c or d position and students ignored the number in the c or d position and entered the sum of a and b; complete the sum, responses when the blank was in the a or b position and students filled in the blank to complete the equation to a number in the c or d position; and a sum of all terms category, when students added or subtracted all the numbers in the equation.

Results and Discussion

All students in this study had teachers who had completed the professional development associated with the New Zealand Numeracy Project (MoE, 2004). The New Zealand project aims to develop student facility to work flexibly with numbers through developing their computational strategies. An espoused intention of the project is to use the structural thinking the students construct as a foundation for understanding algebra and developing early algebraic reasoning (Hannah, 2006). Despite this intended focus, the findings of this current study reveal that 46% of all the students only used arithmetic strategies, 28% of all students only used relational strategies, and 26% of students used a mixture of arithmetic and relational strategies.

How Consistent was the Student's Strategy Use Across the Year Levels?

Table 2 illustrates the distribution of students at each year level in each category. Consistently at every year level from Year 5 to Year 8 students predominantly used arithmetic strategies to solve the open number sentence problems.

The data in the table illustrate that the number of students classified as "stable relational" increased across the year levels. The most significant increase was between the Year 5 and Year 6 level. Increases in use of relational thinking between Year 6 and Year 7 students and Year 7 and Year 8 students were relatively small. The number of students classified as "stable arithmetic" decreased from Year 6, to Year 7, and Year 8 with some corresponding rises in the number of students classified as "stable relational" or "non-stable relational".

Table 2
Percentage of Students at Each Year Level in Each Category

	Stable Arithmetic	Stable Relational	Non-stable relational
Year 5	50	19	31
Year 6	53	27	20
Year 7	46	31	23
Year 8	35	34	31

How Were the Problems Solved by Students Using Relational Thinking?

This section outlines the student responses that represent relational thinking given in response to the open number equivalence problems. Responses in this category indicated that the students were able to identify the relation between each side of the equal sign and use this to solve the problem. They were also able to use the correct direction of variation between the uncalculated equations on each side of the equal sign to solve the problem.

Year 5 student: 73 and 72 are 1 apart leaving 73 as the bigger number so I know that I need to make 72's partner 1 bigger than 49.

Year 5 student: 41 is two more than 39 so I have to take away 2 more to make the same answer.

Year 6 student: 48 is 5 more than 43. To make it fair the number in the box has to be 5 bigger than 76.

Year 7 student: If 99 is 9 more than 90, you would need 9 more than 59 to equal it out.

Year 8 student: Subtraction is different to addition. You have to add the 2 on to the first number, you also have to add it on the second to get the same answer 39 + 2 = 41 so you have to add two on to the other number 15 + 2 = 17.

What Were the Common Errors Students Made When Solving the Open Number Equivalence Problems?

A range of student errors were identified. Many of these errors were due to miscalculations as the students attempted to solve the problems using computation. A significant number of errors was also made in group B equivalence problems that involved subtraction. These errors were due to students failing to identify the correct direction of variation between the uncalculated equations. The following examples illustrate that the students have not identified the correct direction of variation in the uncalculated subtraction equations.

> Year 5 student: There is 9 between the two numbers so 99's partner needs to be 9 less than 59.
>
> Year 6 student: 46 is 1 bigger than 45 so I minused the 1 from 104 to get 103.
>
> Year 7: I did 13 because it is two less than 15 so 39 – 15 and 41 – 13 would have to have the same answer.

Examination of the data revealed that when the blank space was in specific positions the student responses indicated a lack of understanding of the equal sign. Predominantly across all year levels the students displayed an error identified by Freiman and Lee (2004) that they termed "complete the sum". This error occurred when the blank was in position A or B of an equation such as A + B = C + D. This error suggests that these students viewed the number on the right of the equal sign as providing the answer. The data in Table 3 shows the percentage of students at each year level demonstrating this error in their response. Responses showing this error remained consistent across students from Year 5 to Year 7 but decreased at Year 8 level.

Table 3
Percentage of Student Responses Which Were Classified as the "Complete the Sum" Error

	Year 5	Year 6	Year 7	Year 8
43 + **5** = 48 + 76 43 + **33** = 48 + 76	16	19	15	3
7 + 17 = 15 + 24	3	6	7	0
99 – **9** = 90 – 59	14	17	14	3

Freiman and Lee (2004) identified a common student error they termed "direct sum". As illustrated in the data when the blank space was in position C or D the students treated the equivalence problem as a direct sum. In this case they ignored the other number and put the answer to A plus B in the blank space. This error suggests that these students view the equal sign as an indication to write the answer. The data in Table 4 reveal the percent of students making this error decreased slightly over the year levels. However it should be noted that this was the most common error still occurring in the Year 8 students' responses.

Table 4
Percentage of Student Responses Which Were Classified as the "Direct Sum" Error

	Year 5	Year 6	Year 7	Year 8
23 + 15 = 26 + **38**	0	2	7	1
104 – 45 = **59** – 46	3	11	11	5
39 – 15 = 41 – **24**	8	6	5	5

Students also made errors of adding all the numbers in the equivalence problem and putting their sum in the blank space. Freiman and Lee (2004) label this error as "sum of all terms". This error indicates students have over-generalized the property of addition and have ignored the importance of the sequence of symbols in the problem. The data in Table 5 displays the percentage of students at each year level showing this error in their responses. This error was less commonly made by students at the higher year levels.

Table 5
Percentage of Student Responses Which Were Classified as the "Sum of all Terms" Error

	Year 5	Year 6	Year 7	Year 8
73 + 49 = 72 + **194**	8	6	1	1
23 + 15 = 26 + **64**	8	6	3	2
104 – 45 = **13** – 46	3	4	7	3

Conclusion and Implications

This study sought to examine students' use of strategies to solve open number equivalence problems. The results indicate that relatively few students made consistent use of relational strategies across the different sets of equivalence problems. In contrast, many students consistently used arithmetic strategies across all the sets of equivalence problems. However the results also showed that the number of students using only arithmetic strategies decreased across the year levels with more students making some use of relational strategies in combination with arithmetic strategies.

All students within this study had been involved in a mathematics program that focused on strengthening their use of efficient computational strategies for the past three years. However, although emphasis had been placed on developing a flexible range of strategies, many students demonstrated an inability or disinclination to use relational thinking to solve the equivalence problems. These results highlight a need to balance teaching of computational strategies with explicit attention to the fundamental concepts of algebraic reasoning such as relational thinking.

Examination of common student errors when solving the equivalence problems also highlighted some students' lack of understanding of the equal sign. Errors the students made reflected their view of the equal sign as an indication to carry out an operation. Although many of these errors occurred more frequently in the earlier year levels, the frequency of these errors occurring in Year 7 is of some concern, as is the persistence of the "direct sum" error in Year 8. These results support other researchers' contention that greater attention needs to be paid to developing students' understanding of the equal sign through primary and middle school (Carpenter et al., 2003; Falkner et al., 1999; Knuth et al., 2006).

Implications of this study would suggest that an emphasis on increasing numerical reasoning is not adequate to develop deep powerful understandings of essential algebraic concepts. To develop students' algebraic reasoning, explicit attention needs to be given to developing relational forms of thinking. This also requires focus on developing students' notions of the equal sign as representing relational equivalence.

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